Exam 2 will be held during lecture on Wednesday 15-apr
Review the schedule (most recent 24-mar) to see the sections
Exam 1 topics are foundational only
Look for review packet* by Sat -- meantime, review lecture notes, relevant homework
*will contain *additional* practice problems

§ 5.1

$n \times m$ matrix

#rows #columns

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & \ddots & & \\ \vdots & & a_{ij} & \\ a_{n1} & \cdots & \cdots & a_{nm} \end{bmatrix}$$

upper case

denote entry in the ith row, jth column

recall det A given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$

<u>matrix addition</u> - add each corresponding entry

given $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$ then $A + B = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$

multiply a matrix by scalar, k $\quad k \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} k & 2k \\ 3k & 4k \end{bmatrix}$

real constant

<u>zero matrix</u> - all entries are zero ex $M = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

matrix with a single column considered a <u>column vector</u>

$\vec{b} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ or $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

lower case

3x1

similarly row vector format: ex. $\vec{a} = \begin{bmatrix} -5 & 2 & 0 \end{bmatrix}$

multiplication

matrix times vector:

given $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ then $A\vec{x} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{cases} 1x_1 + 2x_2 \\ 3x_1 + 4x_2 \end{cases}$

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think of a system of eqns with matrix A being the set of coefficients

$$\text{ex.} \quad \overset{A}{\begin{bmatrix} 2 & -3 & 1 \\ 4 & 5 & -2 \\ 6 & -7 & 0 \end{bmatrix}} \overset{\vec{x}}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \begin{bmatrix} 2x - 3y + z \\ 4x + 5y - 2z \\ 6x - 7y \end{bmatrix}$$

3×3 $\quad$ 3×1

result will be 3×1

A is a coefficient matrix

ex. given $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$ $\vec{x} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

2×3 $\quad$ 3×1 $\quad$ result 2×1

$$A\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \underset{\text{scalar}}{3} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} 1 + \begin{bmatrix} -1 \\ 3 \end{bmatrix} 2$$

$$= \begin{bmatrix} 3 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

<u>identity matrix</u> - square $n \times n$ where diagonal entries are 1 and all other entries are zero

n is order

ex $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad I_n = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$

diagonal

$AI = A = IA$ $\quad$ assuming A is square and I is same order

<u>Differentiating a Matrix</u>

given $A(t)$ is a matrix

then $A'(t) = \frac{d}{dt} A = \frac{dA}{dt}$ $\longleftarrow$ differentiate each entry

ex. given $\vec{x} = \begin{bmatrix} t \\ t^2 \\ e^{-t} \end{bmatrix}$ $\quad (\vec{x})' = \begin{bmatrix} 1 \\ 2t \\ e^{-t} \end{bmatrix}$

ex. given $\vec{x} = \begin{bmatrix} t \\ t^2 \\ e^{-t} \end{bmatrix}$ $\quad (\vec{x})' = \begin{bmatrix} 1 \\ 2t \\ -e^{-t} \end{bmatrix}$

$\vec{x}(t)$

ex. given $A = \begin{bmatrix} \sin t & 1 \\ t & \cos t \end{bmatrix}$ then $\frac{dA}{dt} = \begin{bmatrix} \cos t & 0 \\ 1 & -\sin t \end{bmatrix}$

derivative rules would apply as you would expect:
given A, B are matrices and c is a constant

terms: $\frac{d}{dt}(A+B) = \frac{d}{dt}A + \frac{d}{dt}B$

product: $\frac{d}{dt}(AB) = \frac{d}{dt}AB + A\frac{d}{dt}B$

scalar: $\frac{d}{dt}(cA) = c\frac{d}{dt}A$

capital

if C is a constant matrix $\frac{d}{dt}(CA) = C\frac{d}{dt}A$

<u>1ST Order Linear Systems</u> general sys of n 1ST order eqns:

$$(x_1)' = p_{11}(t)x_1 + p_{12}(t)x_2 + \dots + p_{1n}(t)x_n + f_1(t)$$
$$(x_2)' = p_{21}(t)x_1 + p_{22}(t)x_2 + \dots + p_{2n}(t)x_n + f_2(t)$$
$$\vdots$$
lower case
$$(x_n)' = p_{n1}(t)x_1 + p_{n2}(t)x_2 + \dots + p_{nn}(t)x_n + f_n(t)$$

lower case

from this, create a coefficient matrix: $P(t) = \left[p_{ij}(t)\right]$

upper case

then visualize each column as a column vector in the form:
$\vec{x} = [x_i]$ and $\vec{f}(t) = [f_i(t)]$

now system of equations CBW as $(\vec{x})' = P(t)\vec{x} + \vec{f}(t)$

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with a sol'n on open I being a column vector $\vec{x}(t) = [x_i(t)]$

s.t. the component functions of $\vec{x}$ satisfy the system

ex. given 1st order system:

$$(x_1)' = 4x_1 - 3x_2$$
$$(x_2)' = 6x_1 - 7x_2$$

rewrite as a matrix:

$$\frac{d}{dt}\vec{x} = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$(\vec{x})' = P \quad \vec{x}$$

verify that $\vec{x}_1(t) = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}$ is a solution and

$\vec{x}_2(t) = \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}$ is also a solution

vector functions

for $\vec{x}_1(t)$: $P\vec{x}_1 = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix} = \begin{bmatrix} 12e^{2t} - 6e^{2t} \\ 18e^{2t} - 14e^{2t} \end{bmatrix}$

derivatives of $\vec{x}_1$

$$= \begin{bmatrix} 6e^{2t} \\ 4e^{2t} \end{bmatrix} = \begin{bmatrix} 2 \cdot 3e^{2t} \\ 2 \cdot 2e^{2t} \end{bmatrix} = (\vec{x}_1)'$$

verify for $x_2(t)$ on your own

consider $\frac{d\vec{x}}{dt} = P(t)\vec{x}$ as equivalent to HG system or $\vec{f}(t) \equiv 0$

expect system to have n solutions $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$

that are linearly independent and CBW as a linear combination of the n particular solutions

<u>Principle of Superposition</u> if $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ are n solutions of a HG linear eq'n $\frac{d}{dx}\vec{x} = P(t)\vec{x}$ on open I and $c_1, c_2, \ldots, c_n$ are constants then

linear combination:

and $c_1, c_2, \ldots, c_n$ are constants then linear combination:

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \ldots + c_n\vec{x}_n(t)$$

is also a solution

revisit previous ex:

if $\vec{x}_1$ and $\vec{x}_2$ are 2 solutions of $\frac{d}{dx}\vec{x} = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix}\vec{x}$

then LC $\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$

$$\vec{x}(t) = c_1\begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix} + c_2\begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}$$ is also a solution

then using $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\vec{x}_1(t) = 3c_1e^{2t} + c_2e^{-5t}$$
$$\vec{x}_2(t) = 2c_1e^{2t} + 3c_2e^{-5t}$$

is gen sol'n

Linear Independence

vectors $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ are linearly dependent if constants $c_1, c_2, \ldots, c_n$ are not all zero s.t. $c_1\vec{x}_1 + c_2\vec{x}_2 + \ldots + c_n\vec{x}_n = 0$

linearly independent when none are a LC of other

as seen previously, use Wronskian to determine if eqns in a system are linearly independent

$$W(t) = \begin{array}{c} \begin{array}{cccc} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{array} \\ \begin{vmatrix} x_{11}(t) & x_{12}(t) & \cdots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & & \\ \vdots & & \ddots & \\ x_{n1}(t) & & & x_{nn}(t) \end{vmatrix} \end{array}$$

nxn determinant